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Some remarks on p-th compounds of nonsingular matrix

Introduction. Let $A = \| a^i_j \|$ be a nonsingular n-by-n matrix. We consider the set of p-by-p minors of the matrix A

(1) $$A\begin{pmatrix} l_1, \dots, l_p \\ k_1, \dots, k_p \end{pmatrix}. \quad 1 \leqslant \begin{matrix} l_1 < l_2 <, \dots, l_p \\ k_1 < k_2 <, \dots, k_p \end{matrix} \leqslant n.$$

The number of these minors (1) is equal to N^2, where $N = \binom{n}{p}$. We denote by

$$A_p = \| A^{\alpha}_{\beta} \|$$

the p-th compounds of the matrix A, where

$$A^{\alpha}_{\beta} = A\begin{pmatrix} l_1, & \dots, & l_p \\ k_1, & \dots, & k_p \end{pmatrix}$$

(cf. [1], p. 30).

It is known (cf. [1], p. 31) that the mapping

(2) $$h_p : GL(n, R) \to GL(N, R)$$

determined by

(3) $$h_p(A) = A_p$$

is a homomorphism of these groups.

In this note the following theorems will be proved:

THEOREM 1. *The determinant of matrix A_p is given by the formula*

$$\operatorname{Det} A_p = (\operatorname{Det} A)^{\binom{n-1}{p-1}}.$$

THEOREM 2. *The kernel of the homomorphism h_p, $1 \leqslant p \leqslant n-1$, is equal to the subgroup:*

$$\operatorname{Ker} h_p = \begin{cases} \{-E_n, E_n\} & \text{for p-even,} \\ \{E_n\}, & \text{for p-odd,} \end{cases}$$

where E_n is te n-by-n unit matrix.

§ 1. **Proof of theorem 1:** We denote by D_p the determinant of the matrix A_p. From theorem 1.2 (cf. [2], p. 3) it follows that

$$D = \varphi\,(\mathrm{Det}\ A), \tag{4}$$

where the function φ fulfils the equation

$$\varphi\,(\xi\,\eta) = \varphi\,(\xi)\,\varphi\,(\eta)\,.$$

Here ξ, η are non-zero real numbers.

Let us put in (4) instead of A the diagonal matrix

$$B = \| b^i_k \| = \| a^i \delta^i_k \|,\ a^1 = a \neq 0,\ a^2 = a^3 =, \ldots, = a^n = 1. \tag{5}$$

The matrix $h_p\,(B)$ is a diagnal matrix with the elements

$$a^{i_1} a^{i_2} \ldots a^{i_p}\ 1 \leqslant i_1 <, \ldots, < i_p \leqslant n,$$

where

$$a^{i_1} \ldots a^{i_p} = \begin{cases} a, & \text{for } i_1 = 1 \\ 1, & \text{for } i_1 \neq 1\,. \end{cases}$$

The number of elements with $i_1 = 1$ is equal to $\binom{n-1}{p-1}$.

Hence we obtain

$$D_p = a^{\binom{n-1}{p-1}}. \tag{6}$$

Putting $a = \mathrm{Det}\ A$ into (6) we have

$$D_p = (\mathrm{Det}\ A)^{\binom{n-1}{p-1}}. \tag{7}$$

This completes the proof.

Proof of theorem 2. We show that Ker h_p, $1 \leqslant p \leqslant n-1$, is contained tn the centre of the group $GL\,(n, R)$ The unimodular matrix $A \in GL\,(n, R)$, where

$$A = \left\| \begin{matrix} 1 & 0 & & 0 \\ 1 & 1 & & 0 \\ & & & \\ & & & 0 \\ 0 & & 1 & 1 \end{matrix} \right\|$$

is not contained in the Ker h_p, $1 \leqslant p \leqslant n - 1$.
The matrix $h_p\,(A) = \| A^\alpha_\beta \|$ fulfils the condition

$$\| A^\alpha_\beta \| \neq E_N\,. \tag{8}$$

where E_N denotes unit N-by-N matrix. In fact, for $\alpha_0 = (2, \ldots, p + 1)$ $\beta_0 = (1, \ldots, p)$, $\alpha_0 \neq \beta_0$, we have the equality

$$a^{\alpha_0}_{\beta_0} = A\begin{pmatrix} 2, & 3, \ldots, & p+1 \\ 1, & 2, \ldots, & p \end{pmatrix} = 1. \tag{9}$$

From theorem 2.1 (c. f. [3], p. 212) we conclude that Ker h_p is contained in the center of the group $GL(n, R)$. Then for every $S \in \mathrm{Ker}\ h_p$ we have

$$S = \begin{pmatrix} \alpha & & 0 \\ 0 & & \alpha \end{pmatrix}, \quad \alpha \neq 0 . \tag{10}$$

Now we show that $\alpha = 1$ or $\alpha = -1$, $1 \leqslant p \leqslant n-1$.

From theorem 1. it follows that

$$\mathrm{Det}\ h_p(S) = (\mathrm{Det}\ S)^{\binom{n-1}{p-1}} = \alpha^{n\binom{n-1}{p-1}} = 1, \quad 1 \leqslant p \leqslant n-1. \tag{11}$$

From (1) we have

$$\mathrm{Ker}\ h_p \subset \{-E_n, E_n\} .$$

On the other hand we have

$$h_p(-E_n) = -E_N, \quad h_p(E_n) = E_N, \quad \text{for } p \text{ odd} \tag{12}$$

and

$$h_p(-E_n) = E_N, \quad h_p(E_n) = E_N, \quad \text{for } p \text{ even} . \tag{13}$$

From (12) and (13) it follows that

$$\mathrm{Ker}\ h_p = \begin{cases} \{-E_n, E_n\}, & \text{for } p \text{ even} \\ \{E_n\}, & \text{for } p \text{ odd.} \end{cases}$$

This completes the proof.

The following conclusions follow from theorem 2.

COLLORARY 1. *If the number p is odd,* $1 \leqslant p \leqslant n-1$, *then the homomorphism* $h_p : GL(n, R) \to h_p(GL(n, R))$, *is an isomorphism for every* n.

COLLORARY 2. *If the number p is even,* $1 \leqslant p \leqslant n-1$, *then the image* $h_p(GL(n, R))$ *is isomorphic with the quotient group* $GL(n, R) / \{-E_n, E_n\}$. *In particular, for n odd the image* $h_p(GL(n, R))$ *is isomorphic with* $GL^+(n, R)$, *where* $GL^+(n, R) = \{A \in GL(n, R) : \mathrm{Det}\ A > 0\}$.

REFERENCES

[2] Ф. Р. ГАНТМАХЕР, *Теория матриц*, Москва, 1966.

[2] M. Kucharzewski: *Einige Bemerkungen über die linearen homogenen geometrischen Objekte ester Klasse*, Ann. Pol. Math XIX (1967), 1—12.

[3] M. Kucharzewski, A. Zajtz: *Über die linearen homogenen geometrischen Objekte des Typus [m, n, 1], wo m ⩽ n ist*, Ann. Pol. Math, XVIII (1966), 205—252.

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PEWNE UWAGI O p-tych POTĘGACH NIEOSOBLIWEJ MACIERZY

Streszczenie

Praca dotyczy pewnego naturalnego homomorfizmu h_p grupy $GL(n, R)$ w grupę $GL\left(\binom{n}{p}, R\right)$ gdzie $1 \leqslant p \leqslant n$, określonego przez podwyznaczniki stopnia p macierzy z $GL(n, R)$ (cf. [1], str. 31.)

W nocie udowodniono dwa twierdzenia o homomorfizmie h_p.

TWIERDZENIE 1. *Wyznacznik macierzy* $h_p(A)$, *gdzie* $A \in GL(n, R)$, *jest dany wzorem*

$$\operatorname{Det}(h_p(A)) = \operatorname{Det}(A)^{\binom{n-1}{p-1}}$$

TWIERDZENIE 2. *Jądro homomorfizmu* h_p, $1 \leqslant p \leqslant n-1$, *jest równe:*

$$\operatorname{Ker} h_p = \begin{cases} \{-E_n, E_n\} & \text{dla } n \text{ parzystego.} \\ \{E_n\} & \text{dla } n \text{ nieparzystego.} \end{cases}$$

gdzie E_p *oznacza macierz jednostkową* $n \times n$.

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